# SECTION 10.5: THE COMPARISON TESTS

Geometric series and $p$-series are two classes of series that we can quickly identify and classify as 'convergent' or 'divergent.' In this section, we develop tools which allow us to analyze series that in some sense 'resemble' geometric and $p$-series as the terms march off to infinity.

**EXAMPLE 1:** Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+2k+1}$.

While this is not a $p$-series, we know from the leading term test that as $k \to \infty$, $\displaystyle\frac{1}{k^3+2k+1} \approx \frac{1}{k^3}$.

In other words, as $k \to \infty$, the terms $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+2k+1}$ 'resemble' the terms of the convergent $p$-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$.

It certainly seems that this should be enough to conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+2k+1}$ converges, so let's see why.

Since $k \geq 1$, the terms $\displaystyle\frac{1}{k^3+2k+1}$ and $\displaystyle\frac{1}{k^3}$ are positive. This means the corresponding sequence of partial sums:

$$\left\{\sum_{k=1}^{n} \frac{1}{k^3+2k+1}\right\} \quad \text{and} \quad \left\{\sum_{k=1}^{n} \frac{1}{k^3}\right\}$$

are **increasing** sequences. If we dig a bit deeper, we see that for all $k \geq 1$, $k^3+2k+1 > k^3 > 0$ which means

$$0 < \frac{1}{k^3+2k+1} < \frac{1}{k^3}.$$

Putting all this together, we get: $\displaystyle 0 < \sum_{k=1}^{n} \frac{1}{k^3+2k+1} < \sum_{k=1}^{n} \frac{1}{k^3} \leq \sum_{k=1}^{\infty} \frac{1}{k^3} = \text{a finite number.}$

Hence, $\displaystyle\left\{\sum_{k=1}^{n} \frac{1}{k^3+2k+1}\right\}$ is an increasing sequence which is bounded above so it converges!

The above argument generalizes to the following theorem:

**DIRECT COMPARISON TEST (DCT):** If $\{a_k\}$ and $\{b_k\}$ satisfy $0 < a_k \leq b_k$ for all $k \geq N$, then:

- If $\displaystyle\sum_{k=N}^{\infty} b_k$ **converges** then so does $\displaystyle\sum_{k=N}^{\infty} a_k$.
- If $\displaystyle\sum_{k=N}^{\infty} a_k$ **diverges** then so does $\displaystyle\sum_{k=N}^{\infty} b_k$.

Paraphrased, 'if the larger series converges, so does the smaller; if the smaller series diverges, so does the larger.'

**NOTE:** The second conditional statement listed in the DCT is the 'contrapositive' of the first statement - which is logically equivalent to the first statement. This means in the grand scheme of things, both statements listed are giving us the same information. (Do you see why?)

**EXAMPLE 2: (VIDEO)** Use the DCT to show:

1. the series $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin(k)|}{k^2}$ converges.

   **HINT:** $|\sin(k)| \leq 1$ for all $k$.

2. the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$ diverges.

   **HINT:** For $k \geq 2$, $\ln(k) < k$ so that $\dfrac{1}{k} < \dfrac{1}{\ln(k)}$.

Suppose we wished to investigate the series: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 - k}$.

As $k \to \infty$, the leading term test tells us that $k^2 - k \approx k^2$ so we compare $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 - k}$ to $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$.

We know $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges (it's 'essentially' a $p$-series with $p = 2 > 1$), however, $k^2 - k < k^2$, so $\dfrac{1}{k^2 - k} > \dfrac{1}{k^2}$.

Hence, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2} < \sum_{k=2}^{\infty} \frac{1}{k^2 - k}$ but it's the 'smaller' series which converges here, so the DCT tells us nothing.

The DCT, much like the integral test, has its niche - but the true value of the DCT for us is the following:

**THE LIMIT COMPARISON TEST (LCT):** Suppose $\{a_k\}$ and $\{b_k\}$ satisfy $a_k > 0$ and $b_k > 0$ for all $k \geq N$.

Suppose in addition that $\displaystyle\lim_{k \to \infty} \frac{a_k}{b_k} = L$. Then:

- if $L > 0$ is a finite number, the series $\displaystyle\sum_{k}^{\infty} a_k$ and $\displaystyle\sum_{k}^{\infty} b_k$ either **both** converge or **both** diverge.

- if $L = 0$ and $\displaystyle\sum_{k}^{\infty} b_k$ **converges**, then $\displaystyle\sum_{k}^{\infty} a_k$ **converges** also.

- if $\displaystyle\lim_{k \to \infty} \frac{a_k}{b_k} = \infty$ and $\displaystyle\sum_{k}^{\infty} b_k$ **diverges**, then $\displaystyle\sum_{k}^{\infty} a_k$ **diverges** also.

**NOTE:** In the second and third cases above, we need to know about the series $\displaystyle\sum_{k}^{\infty} b_k$ in the denominator. For that reason, it's best to put the series we know about in the denominator from the start.

**EXAMPLE 3:** Use the LCT to determine if the following series converge or diverge:

1. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 - k}$.

Ans: compare with $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2}$; converges

2. **(VIDEO)** $\displaystyle\sum_{k=1}^{\infty} \frac{3k+1}{k\sqrt{k+1}}$.

Ans: compare with $\displaystyle\sum_{k=1}^{\infty} \frac{3}{\sqrt{k}}$; diverges

3. **(VIDEO)** $\displaystyle\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$.

Ans: compare with $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{1.5}}$; converges

4. **(VIDEO)** $\displaystyle\sum_{k=1}^{\infty} \frac{2^k - k}{3^{2k}}$.

Ans: compare with $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{2k}}$; converges

**HOMEWORK:** Section 10.5: 9 - 63 odd.